ACTION INTEGRALS ALONG CLOSED ISOTOPIES IN COADJOINT ORBITS

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ABSTRACT. Let $\mathcal{O}$ be the orbit of $\eta \in \mathfrak{g}^*$ under the coadjoint action of the compact Lie group G. We give two formulae for calculating the action integral along a closed Hamiltonian isotopy on $\mathcal{O}$. The first one expresses this action in terms of a particular character of the isotropy subgroup of η. In the second one is involved the character of an irreducible representation of G.

1. Introduction

Let (M, ω) be a quantizable [13] symplectic manifold. We will denote by $\text{Ham}(M)$ the group of Hamiltonian symplectomorphisms [8] of (M, ω). In this note we will consider loops $\{\psi_t\}_{t \in [0,1]}$ in $\text{Ham}(M)$ at id. Given $q \in M$, the loop ψ generates the closed curve $\{\psi_t(q) \,|\, t \in [0,1]\}$ in M which is homologous to zero [8, page 334]. As (M, ω) is quantizable, it makes sense to define the action integral $\mathfrak{A}_\psi(q)$ along such a curve as the element of $\mathbb{R}/\mathbb{Z}$ given by the formula [11] [8]

$$\mathfrak{A}_\psi(q) = \int_S \omega - \int_0^1 f_t(\psi_t(q))dt + \mathbb{Z}, \tag{1.1}$$

where S is any 2-surface whose boundary is the curve $\{\psi_t(q)\}$, and where f_t a fixed time dependent Hamiltonian associated to $\{\psi_t\}$.

Since (M, ω) is quantizable, one can choose a prequantum bundle L on M, endowed with a connection D [13]. On the other hand, let X_t be the corresponding Hamiltonian vector fields determined by f_t, then one can construct the operator $\mathcal{P}_t := -D_{X_t} - 2\pi i f_t$, which acts on the sections of L. The equation $\dot{\tau}_t = \mathcal{P}_t(\tau_t)$ defines a "transport" of the section $\tau_0 \in C^\infty(L)$ along ψ_t. This transport enjoys the following nice property: If $D_Y \tau_0 = 0$, with Y a vector field on M, then $D_{Y_t}\tau_t = 0$, for

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$Y_t = \psi_t(Y)$ (see [10]). From this fact one can prove that τ_1 and τ_0 differ in a constant factor $\kappa(\psi)$; that is, $\tau_1 = \kappa(\psi)\tau_0$. A direct calculation shows that $\kappa(\psi) = \exp(2\pi i \mathfrak{A}_\psi(q))$, where q is an arbitrary point of M [10]. Consequently the expression (1.1) is independent of q, and it makes sense to define the action integral along ψ by (1.1).

The purpose of this note is to calculate the value of the invariant $\kappa(\psi)$ when the manifold M is a coadjoint orbit [5] of a compact Lie group. However in Section 2 we study a more general situation. If a Lie group G acts on the manifold M by symplectomorphisms and there is a moment map for this action, then each $A \in \mathfrak{g}$ determines a vector field X_A on M and the corresponding Hamiltonian f_A. We can construct the respective operator $\mathcal{P}_A$ on $C^\infty(L)$, so one has a representation $\mathcal{P}$ of the Lie algebra $\mathfrak{g}$ on $C^\infty(L)$. When this representation extends to an action ρ of the group G, the prequantization is said to be G-invariant. In this case we will prove that the value of $\kappa(\psi)$ can be expressed in terms of ρ. More precisely, if the isotopy ψ_t is determined by vector fields of type X_{A_t} we show that $\tau_t = \rho(h_t)\tau_0$, where h_t is the solution to Lax equation $\dot{h}_t h_t^{-1} = A_t$.

Section 3 is concerned with the invariant $\kappa(\psi)$ for closed isotopies ψ in a coadjoint orbit of a compact Lie group G. We study the value of $\kappa(\psi)$, when the isotopy is defined by vector fields of type X_{A_t}. Given $\eta \in \mathfrak{g}^*$, the orbit $\mathcal{O}_\eta$ of η admits a G-invariant prequantization if the prequantum bundle is defined by a character Λ of G_η, the subgroup of isotropy of η. In this case we prove that $\kappa(\psi) = \Lambda(h_1)$, with h_t the solution to the corresponding Lax equation (Theorem 6).

If G is a semisimple group, the choice of a maximal torus T contained in G_η permits us to define a G-invariant complex structure on $G/G_\eta = \mathcal{O}_\eta$. This complex structure, in turn, determines a holomorphic structure on L. When the prequantization is G-invariant, $\mathcal{P}$ defines also a representation ρ of G on the space $H^0(L)$ of holomorphic sections of L. When G_η itself is a maximal torus, the Borel-Weil theorem allows us to characterize ρ in terms of its highest weight. We prove that the invariant $\kappa(\psi)$ for the closed isotopy considered above is equal to $\chi(\rho)(h_1)/\dim\rho$. This fact permits us to calculate $\kappa(\psi)$ using the Weyl's character formula. This stuff is considered in Section 4.

In Section 5 we check the results of Sections 3 and 4 in two particular cases. In the first one we calculate directly the value of $\kappa(\psi)$ for a closed isotopy ψ in $\mathbb{CP}^1$; Theorem 6 and Weyl's character formula applied to this example give the same result as the direct calculation. In [10] we determined the value $\kappa(\psi)$ for a closed Hamiltonian flow ψ in S^2; here we recover this number by applying Theorem 6 to this isotopy.

2. G-invariant prequantum data.

Let G be a compact Lie group which acts on the left on the symplectic manifold (M, ω) by symplectomorphisms. We assume that this action is Hamiltonian, and that $\Phi : M \to \mathfrak{g}^*$ is a map moment for this action.

Given $A \in \mathfrak{g}^*$, we denote by X_A, the vector field on M generated by A. Then $(d\Phi(Y)) \cdot A = \omega(Y, X_A)$, for any vector field Y on M. The $\mathbb{R}$-valued map $\Phi \cdot A$ will be denoted by f_A; so

$$\iota_{X_A}\omega = -df_A \quad \text{and} \quad \{f_A, f_B\} = \omega(X_B, X_A) = f_{[A,B]}. \tag{2.1}$$

As we said one assumes that (M, ω) is quantizable. Let L be a prequantum bundle, i.e. L is a Hermitian line bundle over M with a connection D, whose curvature is $-2\pi i\omega$, then one can define the prequantization map [9]

$$A \in \mathfrak{g} \mapsto \mathcal{P}_A = -D_{X_A} - 2\pi i f_A \in \mathrm{End}(C^\infty(L)). \tag{2.2}$$

Proposition 1. *The map $\mathcal{P}$ is a Lie algebra homomorphism.*

Proof. Since the action of G is on the left, the map $A \in \mathfrak{g} \mapsto X_A \in \Xi(M)$, where $\Xi(M)$ denotes the set of vector fields on M, is a Lie Algebra antihomomorphism (see [6] p.42); that is,

$$X_{[A,B]} = -[X_A, X_B] \tag{2.3}$$

On the other hand, if τ is a section of L

$$[\mathcal{P}_A, \mathcal{P}_B]\tau = [D_{X_A}, D_{X_B}]\tau + 4\pi i\omega(X_A, X_B)\tau. \tag{2.4}$$

Since the curvature of D is $-2\pi i\omega$

$$-2\pi i\omega(X_A, X_B)\tau = [D_{X_A}, D_{X_B}]\tau + D_{[X_A, X_B]}\tau.$$

Using (2.3), (2.1) and (2.4) one obtains

$$[\mathcal{P}_A, \mathcal{P}_B]\tau = \mathcal{P}_{[A,B]}\tau.$$

□

The prequantum data (L, D) are said to be G-invariant, if there is a action ρ of G on $C^\infty(L)$ which generates $\mathcal{P}$ [4]. Henceforth in this Section we assume that the prequantum data are G-invariant.

Let $\{A_t\}_t$ be a curve in $\mathfrak{g}$ with $A_0 = 0$. Given $\tau \in C^\infty(L)$ we consider the equation for the section τ_t of L

$$\frac{d\tau_t}{dt} = \mathcal{P}_{A_t}(\tau_t), \quad \tau_0 = \tau \tag{2.5}$$

This is the equation of the "transport" of the section τ along the isotopy determined by the vector fields X_{A_t} (see [10]). We will try to find a curve h_t in G, such that $h_0 = e$ and $\rho(h_t)(\tau) = \tau_t$, where τ_t is

solution to (2.5). As $\rho : G \to \mathrm{Diff}(C^\infty(L))$ is a group homomorphism, $\rho \circ \mathcal{L}_g = \mathcal{L}_{\rho(g)} \circ \rho$, where $\mathcal{L}_a$ is the left multiplication by a in the respective group. The corresponding tangent maps satisfy

$$\rho_* \circ \mathcal{L}_{g*} = \mathcal{L}_{\rho(g)_*} \circ \rho_*. \tag{2.6}$$

If we put F_t for diffeomorphism $\rho(h_t) =: F_t$, and we define $Y_t \in \mathfrak{g}$ by

$$\dot{h}(t) = \mathcal{L}_{h(t)_*}(Y_t),$$

then by (2.6)

$$\frac{dF_t}{dt} = \rho_*(\dot{h}_t) = \mathcal{L}_{F_t *}(\mathcal{P}(Y_t)). \tag{2.7}$$

As $\mathcal{L}_{F_t}(C) = F_t \circ C$, if $C \in \mathrm{End}(C^\infty(L)) \subset \Xi(C^\infty(L))$, then (2.7) can be written

$$\frac{dF_t}{dt} = F_t \circ \mathcal{P}(Y_t).$$

If we introduce this formula in (2.5), we obtain

$$\frac{d\tau_t}{dt} = (F_t \circ \mathcal{P}(Y_t))\tau = (\mathcal{P}_{A_t} \circ F_t)\tau.$$

Hence

$$F_t \circ \mathcal{P}_{Y_t} \circ F_t^{-1} = \mathcal{P}_{A_t} \tag{2.8}$$

Let $\{m(u)\}_u$ a curve in G which defines $Y_t \in \mathfrak{g}$, then

$$F_t \circ \mathcal{P}_{Y_t} \circ F_t^{-1} = \frac{d}{du}\bigg|_{u=0} \rho(h_t m(u) h_t^{-1}).$$

By (2.8) one can take $Y_t = \mathrm{Ad}_{h_t^{-1}} A_t$; so h_t is the solution to the Lax equation

$$\dot{h}_t h_t^{-1} = A_t \qquad h_0 = e. \tag{2.9}$$

We have proved

Theorem 2. *The solution τ_t to (2.5) is given by $\rho(h_t)\tau$, where h_t satisfies equation (2.9).*

Let $\{A_t \,|\, t \in [0,1]\}$ be a curve in $\mathfrak{g}$ such that the Hamiltonian isotopy $\{\psi_t\}_{t\in[0,1]}$ generated by the vector fields X_{A_t} is closed; i.e. $\psi_0 = \psi_1 = \mathrm{id}$. We have proved in [10] that if τ_t is the solution of (2.5), then

$$\tau_1 = \kappa(\psi)\tau, \tag{2.10}$$

for every $\tau \in C^\infty(L)$, where $\kappa(\psi) = \exp\big(2\pi i \mathfrak{A}_\psi(q)\big)$, and $\mathfrak{A}_\psi(q)$ is the action integral along the curve $\{\psi_t(q)\}_t$, for q arbitrary in M. On the other hand, if h_t is a curve in G solution to (2.9), by Theorem 2 $\tau_1 = \rho(h_1)(\tau)$. It follows from (2.10) that $\rho(h_1) = \kappa(\psi)\,\mathrm{Id}$. Thus we have

Corollary 3. *If W is a finite dimensional ρ-invariant subspace of $C^\infty(L)$, and ρ_W is the restriction of ρ to this subspace, then for the character of ρ_W holds the following formula*

$$\chi(\rho_W)(h_1) = \kappa(\psi)\, dim(W).$$

3. The invariant $\kappa(\psi)$ in a coadjoint orbit

Let G be a compact Lie group, and we consider the coadjoint action of G on $\mathfrak{g}^*$ defined by

$$(g \cdot \eta)(A) = \eta(g^{-1} \cdot A),$$

for $g \in G$, $\eta \in \mathfrak{g}^*$, $A \in \mathfrak{g}$ and $g \cdot A = \mathrm{Ad}_g A$ (see [5] [13]).

If X_A is the vector field on $\mathfrak{g}^*$ determined by A, the map $l_g : \mu \in \mathfrak{g}^* \mapsto g \cdot \mu \in \mathfrak{g}^*$ satisfies

$$(l_g)_*(X_A(\mu)) = X_{g \cdot A}(g \cdot \mu). \tag{3.1}$$

Given $\eta \in \mathfrak{g}^*$, by $\mathcal{O}_\eta =: \mathcal{O}$ will be denoted the orbit of η under the coadjoint action of G. On $\mathcal{O}$ one can consider the 2-form ω determined by

$$\omega_\nu(X_A(\nu), X_B(\nu)) = \nu([A, B]). \tag{3.2}$$

This 2-form defines a symplectic structure on $\mathcal{O}$, and the action of G preserves ω. For each $A \in \mathfrak{g}$ one defines the function $h_A \in C^\infty(\mathcal{O})$ by $h_A(\nu) = \nu(A)$, and for this function holds the formula

$$\iota_{X_A}\omega = dh_A. \tag{3.3}$$

The orbit $\mathcal{O}$ can be identified with G/G_η, where G_η is the subgroup of isotropy of η. The Lie algebra of this subgroup is

$$\mathfrak{g}_\eta = \{A \in \mathfrak{g} \,|\, \eta([A, B]) = 0, \text{ for every } B \in \mathfrak{g}\}$$

The orbit $\mathcal{O}$ possesses a G-invariant prequantization iff the linear functional

$$\lambda : C \in \mathfrak{g}_\eta \mapsto 2\pi i \eta(C) \in i\mathbb{R} \tag{3.4}$$

is integral; i. e., iff there is a character $\Lambda : G_\eta \to U(1)$ whose derivative is the functional (3.4) (see [7]). Henceforth we assume the existence of such a character Λ. The corresponding prequantum bundle L over $\mathcal{O} = G/G_\eta$ is defined by $L = G \times_\Lambda \mathbb{C} = (G \times \mathbb{C})/\simeq$, with $(g, z) \simeq (gb^{-1}, \Lambda(b)z)$, for $b \in G_\eta$.

Each section σ of L determines a Λ-equivariant function $s : G \to \mathbb{C}$ by the relation

$$\sigma(gG_\eta) = [g, s(g)]. \tag{3.5}$$

The $\mathbb{C}^\times$-principal bundle associated to L is $L^\times = L - \{\text{zero section}\}$. If v denotes the element $[e, 1] \in L^\times$, then $T_v(L^\times) \simeq (\mathfrak{g} \oplus \mathbb{C})/\mathfrak{f}_v$, with

$$\mathfrak{f}_v = \{(B, \, -2\pi i \eta(B) \,|\, B \in \mathfrak{g}_\eta\}.$$

The connection form Ω on $L^\times$ is constructed in [7] p.198. The form Ω can be written $\Omega = (\theta, \, d)$, where θ is the left invariant form on G whose value at e is η, and $d \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is defined by $d(z) = (2\pi i)^{-1} z$. It is clear that Ω_v vanishes on $\mathfrak{f}_v$ and that it defines an element of $T^*_v(L^\times)$.

On the other hand the section σ determines a lift $\sigma^\sharp : L^\times \to \mathbb{C}$ by the formula

$$\sigma(\pi(y)) = \sigma^\sharp(y) y, \tag{3.6}$$

here $\pi : L \to \mathcal{O}$ is the projection map. It follows from (3.5)

$$s(g) = \sigma^\sharp([g, z]) z. \tag{3.7}$$

We denote by $\mathcal{E}_\Lambda$ the space of Λ-equivariant functions on G. The identification $C^\infty(L) \sim \mathcal{E}_\Lambda$ allows us to translate the action $\mathcal{P}$ defined in (2.2) to a representation of $\mathfrak{g}$ on $\mathcal{E}_\Lambda$.

Theorem 4. *The action $\mathcal{P}$ on $\mathcal{E}_\Lambda$ is given by $\mathcal{P}_A(s) = -R_A(s)$, where R_A is the right invariant vector field on G determined by A.*

Proof. Let σ be a section of L, by (3.3) $\mathcal{P}_A(\sigma) = -D_{X_A}\sigma + 2\pi i h_A \sigma$. We will determine the lift $(\mathcal{P}_A(\sigma))^\sharp$.

The vector $X_A(g \cdot \eta) \in T_{g\cdot\eta}(\mathcal{O})$ is defined by the curve $u \mapsto e^{uA} g \cdot \eta$ in $\mathcal{O}$. A lift of this curve at the point $[g, z] \in L^\times$ will be a curve of the form $\gamma(u) = [e^{uA}g, \, z_u]$, with $z_u = z e^{ux}$. The vector tangent to γ at $[g, z]$ is $\dot\gamma(0) = [R_A(g), x]$, where $R_A(g)$ is the value at g of the right invariant vector field in G defined by A.

The condition $\Omega(\dot\gamma(0)) = 0$ implies

$$x = -2\pi i \eta(g^{-1} \cdot A). \tag{3.8}$$

Therefore the horizontal lift of $X_A(g \cdot \eta)$ is

$$X_A^\sharp([g, z]) = [R_A(g), \, -2\pi i \eta(g^{-1} \cdot A)],$$

and by (3.7) the action of $X_A^\sharp([g, z])$ on the function $\sigma^\sharp$ can expressed in terms of s

$$X_A^\sharp([g, z])(\sigma^\sharp) = \frac{d}{du}\bigg|_{u=0} \Big(\frac{s(e^{uA}g)}{z e^{ux}} \Big) = \frac{R_A(g)(s)}{z} - \frac{x s(g)}{z}.$$

Since $X_A^\sharp(\sigma^\sharp) = (D_{X_A}\sigma)^\sharp$ [6, page 115], from (3.8) and (3.7) it turns out that the equivariant function associated to $D_{X_A}\sigma$ is

$$g \in G \mapsto R_A(g)(s) + 2\pi i \eta(g^{-1} \cdot A) s(g) \in \mathbb{C}. \tag{3.9}$$

Obviously the equivariant function defined by the section $h_A\sigma$ is the function $\lambda_A s$, where $\lambda_A(g) = h_A(gG_\eta) = (g\cdot\eta)(A) = \eta(g^{-1}\cdot A)$. It follows from (3.9) that the equivariant function which corresponds to $-D_{X_A}\sigma + 2\pi i h_A\sigma$ is $-R_A(s)$. □

Corollary 5. *The action $\mathcal{P}$ on $\mathcal{E}_\Lambda$ is induced by the action*

$$\rho : (b, s) \in G \times \mathcal{E}_\Lambda \mapsto s \circ \mathcal{L}_{b^{-1}} \in \mathcal{E}_\Lambda,$$

where $\mathcal{L}_c$ is left multiplication by c in the group G.

Proof. If $g_t = e^{tA} \in G$, then

$$\frac{d\,\rho_{g_t}(s)}{dt}\bigg|_{t=0}(g) = \frac{d}{dt}\bigg|_{t=0} s(e^{-tA}g) = -R_A(g)(s) = \mathcal{P}_A(s)(g).$$

□

From Corollary 5 it follows that the prequantum data (L, D) are G-invariant.

Let $\{\psi_t \,|\, t \in [0,1]\}$ be a closed Hamiltonian isotopy on $\mathcal{O}$; that is, a Hamiltonian isotopy such that $\psi_1 = \mathrm{id}$. We also assume that the corresponding Hamiltonian vector fields are invariant; that is,

$$\frac{d\psi_t(q)}{dt} = X_{A_t}(\psi_t(q)), \text{ with } A_t \in \mathfrak{g}.$$

If σ is a section of L, σ_t will denote the solution to the equation

$$\frac{d\sigma_t}{dt} = \mathcal{P}_{A_t}(\sigma_t), \qquad \sigma_0 = \sigma. \tag{3.10}$$

By Theorem 4, equation (3.10) on the points $\{g_t\}_{t\in[0,1]}$ of a curve in G gives rise to

$$\dot{s}_t(g_t) = -R_{A_t}(g_t)(s_t), \tag{3.11}$$

for the corresponding equivariant functions. In particular, if g_t is the curve such that $g_0 = e$ and $\dot{g}_t = R_{A_t}(g_t) \in T_{g_t}(G)$; in other words, g_t satisfies the Lax equation $\dot{g}_t g_t^{-1} = A_t$, then

$$R_{A_t}(g_t)(s_t) = \frac{d}{du}\bigg|_{u=t} s_t(g_u).$$

Using (3.11) one deduces

$$\dot{s}_t(g_t) + \dot{g}_t(s_t) = 0 \tag{3.12}$$

If we consider the function $w : [0,1] \to \mathbb{C}$ defined by $w_t = s_t(g_t)$; by (3.12) w is constant. So $s_1(g_1) = s_0(e)$. If $g_1 \in G_\eta$, as s_1 is Λ-equivariant $s_1(g_1) = \Lambda(g_1^{-1})s_1(e)$; so

$$\sigma_1(eG_\eta) = \Lambda(g_1)\sigma_0(eG_\eta). \tag{3.13}$$

The following Theorem is consequence of (2.10) and (3.13)

Theorem 6. *If $\{\psi_t\}$ is the closed Hamiltonian isotopy in $\mathcal{O}$ generated by the vector fields $\{X_{A_t}\}$, then $\kappa(\psi) = \Lambda(g_1)$, where $g_t \in G$ is the solution to $\dot{g}_t g_t^{-1} = A_t$, with $g_0 = e$ and $g_1 \in G_\eta$.*

Remark. Theorem 6 can also be deduced as a consequence of Theorem 2 and Corollary 5. In fact, if h_t is the solution to $\dot{h}_t h_t^{-1} = A_t$, with the introduced notations

$$\sigma_1(a) = (\rho(h_1)\sigma)(a) = [a, s(h_1^{-1}a)] = \Lambda(h_1)[a, s(a)] = \Lambda(h_1)\sigma(a).$$

4. Relation with Weyl's character formula.

Let us assume that G is semisimple Lie group [2], and let T a maximal torus with $T \subset G_\eta$ (see [4] p.166). One has the corresponding decomposition of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ in direct sum of root spaces

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \sum \mathfrak{g}_\alpha,$$

where $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$, and α ranges over the set of roots. This decomposition gives the real counterpart

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in P} \big(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\big) \cap \mathfrak{g},$$

where P is a set of positive roots. We denote by $\alpha^\vee$ the element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(\alpha^\vee) = 2$, while $\beta(\alpha^\vee) \in \mathbb{Z}$ for every root β.

η extends in a natural way to $\mathfrak{g}_{\mathbb{C}}$. If $Y \in \mathfrak{g}_\alpha$, then as $\alpha^\vee \in \mathfrak{g}_\eta$

$$0 = \eta([\alpha^\vee, Y]) = 2\eta(Y).$$

Hence η vanishes on $\sum \mathfrak{g}_\alpha$. If $\eta(\alpha^\vee) \neq 0$, for all root α, then $\mathfrak{g}_\eta = \mathfrak{t}$; in this case η is said to be regular. Henceforth we assume that η is regular.

We define $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where

$$\mathfrak{n} = \sum_{\alpha \in P} \mathfrak{g}_\alpha.$$

Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$, which corresponds to a Borel subgroup B of G.

We have

$$T_\eta(\mathcal{O}) = \mathfrak{g}/\mathfrak{g}_\eta = \sum_{\alpha \in P} \Big(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\Big) \cap \mathfrak{g}.$$

Hence

$$T_\eta^{\mathbb{C}}(\mathcal{O}) = \sum_{\alpha \in P} \Big(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\Big).$$

One defines

$$T^{0,1}_{\eta}\mathcal{O} := \mathfrak{n},$$

and

$$T^{0,1}_{g\cdot\eta}\mathcal{O} := \{X_{g\cdot A}(g\cdot\eta) \,|\, A\in\mathfrak{n}\}.$$

If $g_1\cdot\eta = g_2\cdot\eta$, then $g_1^{-1}g_2\in T$. As $\mathfrak{g}_\alpha$ is an eigenspace for the action of T, then $g_1^{-1}g_2\cdot A\in\mathfrak{n}$, if $A\in\mathfrak{n}$. Therefore the spaces $T^{0,1}_{g\cdot\eta}$ are well-defined.

For $A\in\mathfrak{n}$, one can define the vector field $\mathcal{A}$ on $\mathcal{O}$ by $\mathcal{A}(g\cdot\eta) = X_{g\cdot A}(g\cdot\eta)$. By (3.1) $(l_g)_*\mathcal{A} = \mathcal{A}$, hence the above complex foliation defined on $\mathcal{O}$ is G-invariant. Since the vector $X_{g\cdot A}(g\cdot\eta)$ is defined by the curve $e^{tg\cdot A}g\cdot\eta = ge^{tA}\cdot\eta$, then the left invariant vector field L_A on G/T is the field which corresponds to $\mathcal{A}$, in the identification of G/T with $\mathcal{O}$.

The vector spaces $T^{1,0}$ are defined in the obvious way. As $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$, the decomposition $T^{\mathbb{C}}(\mathcal{O}) = T^{1,0}\oplus T^{0,1}$ define a complex structure on $\mathcal{O}$. This complex manifold can be identified with $G_{\mathbb{C}}/B$.

We assume that the integral functional λ in (3.4) satisfies $\lambda(\alpha^\vee)\leq 0$ for every $\alpha\in P$; this means that $-\lambda$ is a dominant weight for T [1]. Using the complex structure on $\mathcal{O} = G/T$ and the covariant derivative D on the prequantum bundle $L = G\times_\Lambda\mathbb{C}$, it is possible to define a holomorphic structure in L. The section τ of L is said to be holomorphic iff $D_Z\tau = 0$ for any vector field Z of type $(0,1)$. In this way L can be regarded as a holomorphic line bundle over $G_{\mathbb{C}}/B$. The homomorphism $\Lambda : T\to U(1)$ extends trivially to B, since B is a semidirect product of $H = T_{\mathbb{C}}$ and the nilpotent subgroup whose Lie algebra is $\mathfrak{n}$; and each section σ of L determines a function $s : G_{\mathbb{C}}\to\mathbb{C}$ which is Λ-equivariant.

Given $A\in\mathfrak{n}$, the Proof of Theorem 4 shows that the equivariant function associated to $D_{\mathcal{A}}\sigma$ is the map

$$g\in G_{\mathbb{C}}\mapsto R_{g\cdot A}(g)(s) + 2\pi i\eta(g^{-1}g\cdot A)s(g)\in\mathbb{C}.$$

As η vanishes on $\mathfrak{n}$ and $R_{g\cdot A}(g) = L_A(g)$, the function associated to $D_{\mathcal{A}}\sigma$ is $L_A(s)$. The section σ is holomorphic if $D_{\mathcal{A}}\sigma = 0$, for every $A\in\mathfrak{n}$; in this case $L_A(s) = 0$ for $A\in\mathfrak{n}$, that is, s is a holomorphic function on $G_{\mathbb{C}}$. So the space $H^0(G_{\mathbb{C}}/B, L)$ is isomorphic to the space

$$\mathcal{E}_{\Lambda,P} := \{s : G_{\mathbb{C}}\to\mathbb{C} \,|\, s \text{ is holomorphic and } \Lambda-\text{equivariant}\}.$$

The Borel-Weil Theorem asserts that the action of G on the space $\mathcal{E}_{\Lambda,P}$ given by $g\star s = s\circ\mathcal{L}_{g^{-1}}$ is an irreducible representation of G; more precisely the contragredient representation of that one whose highest weight is $-\lambda$ (see [1] pages 290, 300).

Lemma 7. *If $A \in \mathfrak{n}$, then $[\mathcal{A}, X_B] = 0$ for any $B \in \mathfrak{g}_{\mathbb{C}}$.*

Proof. The flow φ_t determined by X_B is given $\varphi_t(g \cdot \eta) = e^{tB} g \cdot \eta$. And the flow ϕ_t of $\mathcal{A}$ is $\phi_t(g \cdot \eta) = e^{tg \cdot A} g \cdot \eta = g e^{tA} \cdot \eta$. Hence

$$(\varphi_t \circ \phi_t)(g \cdot \eta) = e^{tB} g e^{tA} \cdot \eta = (\phi_t \circ \varphi_t)(g \cdot \eta).$$

□

Proposition 8. *If $D_{\mathcal{A}} \sigma = 0$ for any $A \in \mathfrak{n}$, then $D_{\mathcal{A}} \mathcal{P}_B \sigma = 0$ for any $B \in \mathfrak{g}$.*

Proof. Since $D_{\mathcal{A}} \sigma = 0$, it follows from (2.2)

$$D_{\mathcal{A}}(\mathcal{P}_B \sigma) = -D_{\mathcal{A}} D_{X_B} \sigma + 2\pi i \mathcal{A}(h_B) \sigma. \tag{4.1}$$

As

$$[D_{\mathcal{A}}, D_{X_B}] \sigma = D_{[\mathcal{A}, X_B]} \sigma - 2\pi i \omega(\mathcal{A}, X_B) \sigma,$$

from (4.1) and (3.3) we deduce

$$D_{\mathcal{A}}(\mathcal{P}_B \sigma) = -D_{[\mathcal{A}, X_B]} \sigma.$$

Now the proposition is consequence of Lemma 7. □

A direct consequence of Proposition 8 is

Corollary 9. *$\mathcal{P}$ defines a representation of $\mathfrak{g}$ on $H^0(G_{\mathbb{C}}/B, L)$.*

Denoting by π the irreducible representation of G whose highest weight is $-2\pi i \eta$, and by π^* its dual, we have

Corollary 10. *The representation $\mathcal{P}$ on $H^0(G_{\mathbb{C}}/B, L)$ is the derivative of π^*.*

Proof. It is a consequence of Corollary 5 and Borel-Weil theorem □

The subspace $\mathcal{E}_{\Lambda,P} \subset \mathcal{E}_{\Lambda}$ is invariant under the representation ρ defined in Corollary 5, and the restriction of ρ to $\mathcal{E}_{\Lambda,P}$ is precisely the representation π^*. From by Corollary 3 it follows

Theorem 11. *Let η be an element of $\mathfrak{g}^*$, such that the orbit $\mathcal{O}_\eta$ is quantizable and $-2\pi i \eta$ is a dominant weight for the maximal torus G_η. If $\{\psi_t\}$ is the closed Hamiltonian isotopy in $\mathcal{O}_\eta$ generated by the vector fields $\{X_{A_t}\}$, then*

$$\kappa(\psi) = \frac{\chi(\pi^*)(h_1)}{dim\, \pi}, \tag{4.2}$$

where $h_t \in G$ is the solution to $\dot{h}_t h_t^{-1} = A_t$, $h_0 = e$, and π is the representation of G whose highest weight is $-2\pi i \eta$.

Now the character $\chi(\pi^*)$ and the dimension $\dim \pi$ can be determined by Weyl's character formula [1], and so $\kappa(\psi)$.

5. Examples

The invariant $\kappa(\psi)$ in $\mathbb{CP}^1$. Let G be the group $SU(2)$ and η the element of $\mathfrak{su}(2)^*$ defined by

$$\eta\begin{pmatrix} ci & w \\ -\bar{w} & -ci \end{pmatrix} = -\frac{c}{2\pi}. \tag{5.1}$$

The subgroup of isotropy G_η is $U(1) \subset SU(2)$, so the coadjoint orbit $\mathcal{O}_\eta$ can be identified with $SU(2)/U(1) = \mathbb{CP}^1$. The element

$$g = \begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix} \in SU(2)$$

determines the point $(z_0 : z_1) \in \mathbb{CP}^1$. Hence to $\eta \in \mathcal{O}_\eta$ corresponds $p = (1:0) \in \mathbb{CP}^1$. For $z_0 \neq 0$ we put $(z_0 : z_1) = (1 : x+iy)$.

Denoting by A and B the following matrices of $\mathfrak{su}(2)$

$$A := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.2}$$

by (3.2)

$$\omega_\eta(X_A, X_B) = \eta([A,B]) = \frac{1}{\pi}. \tag{5.3}$$

As

$$e^{tA} = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix},$$

the curve $\{e^{tA}\eta\}$, which defines $X_A(p)$, is $(\cos t : i\sin t)$. Hence $X_A(p)$, expressed in the real coordinates (x,y), is equal to $\left(\frac{\partial}{\partial y}\right)_p$. Similarly $X_B(p) = -\left(\frac{\partial}{\partial x}\right)_p$. Hence it follows from (5.3)

$$\omega_p = \frac{1}{\pi} dx \wedge dy = \frac{i}{2\pi} dz \wedge d\bar{z},$$

where $z = x+iy$. Therefore $(\mathcal{O}_\eta, \omega)$ can be identified with $\mathbb{CP}^1$ endowed with the Fubini-Study form

$$\omega = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2} = \frac{1}{\pi}\frac{1}{(x^2+y^2+1)^2} dx \wedge dy. \tag{5.4}$$

Let us consider the symplectomorphism ψ_t on $\mathbb{CP}^1$ defined by

$$(z_0 : z_1) \in \mathbb{CP}^1 \mapsto (e^{-ia_t} z_0 : e^{ia_t} z_1) \in \mathbb{CP}^1,$$

where $a_t \in \mathbb{R}$. If we assume that $a_0 = 0$ and $a_1 = k\pi$, with $k \in \mathbb{Z}$, then $\{\psi_t \,|\, t \in [0,1]\}$ is a closed Hamiltonian isotopy on $\mathbb{CP}^1$. We will

determine $\kappa(\psi)$ by direct calculation. In real coordinates

$$\psi_t(x,y) = \big(x\cos 2a_t - y\sin 2a_t,\ x\sin 2a_t + y\cos 2a_t\big). \tag{5.5}$$

A straightforward calculation shows that the Hamiltonian vector field X_t defined by

$$\frac{d\psi_t(q)}{dt} = X_t(\psi_t(q))$$

is $X_t(x,y) = 2\dot{a}_t\big(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\big)$. It follows from (5.4)

$$\iota_{X_t}\omega = \frac{-2\dot{a}_t}{\pi(x^2+y^2+1)^2}\big(xdx + ydy\big).$$

A Hamiltonian function f_t associated to X_t is

$$f_t(x,y) = -\frac{\dot{a}_t}{\pi(x^2+y^2+1)} + c_t,$$

c_t being a constant. If we impose $\int f_t\omega = 0$, then

$$c_t = c_t\int_{\mathbb{CP}^1}\omega = \frac{\dot{a}_t}{\pi^2}\int_{\mathbb{CP}^1}\frac{1}{(x^2+y^2+1)^3}dx\wedge dy = \frac{\dot{a}_t}{2\pi}.$$

Thus the normalized Hamiltonian function is

$$f_t(x,y) = \frac{\dot{a}_t}{2\pi}\Big(\frac{x^2+y^2-1}{x^2+y^2+1}\Big).$$

Given $q = (x_0, y_0) \in \mathbb{CP}^1$, from (5.5) it follows that the set

$$\{\psi_t(x_0,y_0)\,|\, t\in[0,1]\}$$

is a circle in the plane (x,y) with centre at $(0,0)$; therefore

$$\int_0^1 f_t(\psi_t(q))dt = \frac{k}{2}\Big(\frac{x_0^2+y_0^2-1}{x_0^2+y_0^2+1}\Big). \tag{5.6}$$

On the other hand the 1-form $\theta = (-x+iy)(x^2+y^2+1)^{-1}(dx+idy)$ satisfies $d\theta = -2\pi i\omega$. And

$$\int_0^1 \theta(X_t)dt = -2k\pi i\frac{x_0^2+y_0^2}{x_0^2+y_0^2+1}. \tag{5.7}$$

From (1.1), (5.6) and (5.7) it follows $\mathfrak{A}_\psi(q) = k/2 + \mathbb{Z}$ and

$$\kappa(\psi) = e^{ik\pi}. \tag{5.8}$$

Next we determine the value of $\kappa(\psi)$ by using the results of Section 3. First of all the prequantum bundle for $(\mathbb{CP}^1, \omega)$ is the hyperplane bundle [3] on $\mathbb{CP}^1$. On the other hand the functional

$$ci \in \mathfrak{u}(1) \subset \mathfrak{su}(2) \mapsto 2\pi i\eta(\mathrm{diag}(ci,\, -ci)) = -ic$$

is the derivative of $\Lambda : g \in U(1) \mapsto g^{-1} \in U(1)$. Therefore the respective prequantum data are $SU(2)$-invariant. The isotopy $\{\psi_t\}$ of $\mathbb{CP}^1$ determines the vector fields X_{A_t}, where $A_t = \mathrm{diag}(-i\dot{a}_t,\, i\dot{a}_t)$. In this case the solution to $\dot{h}_t h_t = A_t$ is $h_t = \mathrm{diag}(e^{-ia_t},\, e^{ia_t})$. Hence, by Theorem 6,

$$\kappa(\psi) = \Lambda(h_1) = h_1^{-1} = e^{ik\pi}.$$

This result agrees with (5.8).

THE INVARIANT κ OF A HAMILTONIAN FLOW IN S^2. For $G = SU(2)$, if

$$\eta : \begin{pmatrix} ai & w \\ -\bar{w} & -ai \end{pmatrix} \in \mathfrak{su}(2) \mapsto \frac{na}{2\pi} \in \mathbb{R},$$

with $n \in \mathbb{Z}$, then the orbit $\mathcal{O}_\eta = SU(2)/U(1) = S^2$ admits and $SU(2)$-invariant quantization and the corresponding character Λ of $U(1)$ is $\Lambda(z) = z^n$.

Let $\tilde{\psi}_t$ be the symplectomorphism of S^2 given by

$$\tilde{\psi}_t(q) = \exp(t(aA + bB)) \cdot q,$$

where $a, b \in \mathbb{R}$ and A, B are the matrices introduced in (5.2) . For $t_1 = (a^2 + b^2)^{-1/2}\pi$, $\tilde{\psi}_{t_1} = \mathrm{id}$; in fact

$$\exp(t(aA + bB)) = \begin{pmatrix} \cos|c| & \epsilon \sin|c| \\ -\bar{\epsilon} & \cos|c| \end{pmatrix}, \tag{5.9}$$

with c=t(b+ai) and $\epsilon = c/|c|$ (see [10]). If we set

$$E := \pi(a^2 + b^2)^{-1/2}(aA + bB),$$

by (5.9) $\exp(E) = -\mathrm{Id}$. So the family $\{\psi_t\}_{t\in[0,1]}$, defined by $\psi_t(q) = \exp(tE)q$, is a closed Hamiltonian flow on the orbit $\mathcal{O}_\eta$. By Theorem 6

$$\kappa(\psi) = \Lambda(e^E) = \Lambda(-\mathrm{Id}) = (-1)^n.$$

This result agrees with that one obtained in [10, Theorem 21] by direct calculation.

This result can be deduced from (4.2), when $n < 0$. Here Lax equation $\dot{h}_t h_t^{-1} = E$ has the solution $h_t = \exp(tE)$. The Weyl's character formula [1] is very simple for the group $SU(2)$; in this case, there is only one positive root α and the Weyl group has only two elements. We take for α the linear map defined by

$$\alpha(\mathrm{diag}(ai,\, -ai)) = 2ai;$$

so $\alpha^\vee = \mathrm{diag}(1,\, -1)$. As we assume that $n < 0$, then $-\lambda := -2\pi i\eta$ is the highest weight of a representation π of $SU(2)$. For $t \in U(1)$,

$t^\lambda = t^{-n}$ and $t^\alpha = t^2$. Therefore (see [1])

$$\dim \pi = -n+1 \text{ and } \chi_\pi(t) = \sum_{k=0}^{-n} t^{-n-2k}.$$

Hence

$$\chi_{\pi^*}(h_1) = \chi_\pi(-1) = (-n+1)(-1)^n,$$

and from Corollary 3 we again obtain the value $(-1)^n$ for $\kappa(\psi)$.

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